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INDECOMPOSABLE CONVEX POLYTOPES

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ABSTRACT

Shephard has given a criterion for the indecomposability (in the sense of Minkowski addition) of a convex polytope, in terms of strong chains of indecomposable faces joining pairs of vertices. Here, this criterion is weakened, to one involving strongly connected sets of indecomposable faces meeting every facet.

1. A criterion for indecomposability

In [3], Shephard gave a criterion for the indecomposability of a convex polytope with respect to Minkowski addition. We recall that a convex polytope P in euclidean space $\mathbf{E}^d$ is said to be *indecomposable* if, in any expression $P = Q + R = \{x + y \mid x \in Q, y \in R\}$, each *summand* Q is homothetic to P or a point, so that $Q = \lambda P + t$ for some $\lambda \geqq 0$ and $t \in \mathbf{E}^d$ (and then $0 \leqq \lambda \leqq 1$ and $R = (1 - \lambda)P - t$). Throughout, we shall follow the terminology of [1].

Shephard's criterion can be phrased succinctly using the concept of a *strong chain* of faces of a polytope P, which is a sequence $F_0, F_1, \ldots, F_k$ of faces such that $\dim(F_{j-1} \cap F_j) \geqq 1$ for $j = 1, \ldots, k$. Such a chain *joins* two vertices u and v of P if (say) $u \in F_0$ and $v \in F_k$. Shephard's result is

THEOREM 1. *A convex polytope, any two of whose vertices can be joined by a strong chain of indecomposable faces, is itself indecomposable.*

A family $\mathcal{F}$ of faces of a polytope P is called *strongly connected* if for each F, $G \in \mathcal{F}$, there exists a strong chain $F = F_0, F_1, \ldots, F_k = G$ with each $F_j \in \mathcal{F}$. A subset $\mathcal{F}$ of faces *touches* a face F of P if $(\bigcup \mathcal{F}) \cap F \neq \emptyset$. Recalling that a *facet* of P is a face F of dimension $\dim F = \dim P - 1$, our new criterion is given by

THEOREM 2. *If a polytope has a strongly connected family of indecomposable faces which touches each of its facets, then it is itself indecomposable.*

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In fact, Theorems 1 and 2 are not strictly comparable; however, in all applications of Theorem 1 which we are aware of, the strong chains fit together to form a strongly connected set.

2. Proof of the theorem

Without loss of generality we consider a d-polytope P in $\mathbf{E}^d$, which possesses a strongly connected family $\mathcal{F}$ of indecomposable faces which touches every facet. We can express P in the form

$$P = \{x \in \mathbf{E}^d \mid \langle x, u_i \rangle \leqq \eta_i \ (i = 1, \dots, n)\},$$

where $u_1, \dots, u_n$ are the outer normal vectors to the facets of P. Now, a summand Q of P is of the same form, with η_i replaced by ζ_i (say) for $i = 1, \dots, n$. In general, the u_i are not all now facet normals of Q; in the present case, however, we must show that they are (unless Q is a point), and, indeed, that $\zeta_i = \lambda\eta_i + \langle t, u_i \rangle$ $(i = 1, \dots, n)$ for some $\lambda \geqq 0$ and $t \in \mathbf{E}^d$, which corresponds to the relation $Q = \lambda P + t$. For further details, see [1], Chapter 14, or [2].

If u is any non-zero vector, we write P_u for the face of P in direction u, that is, the intersection of P with its support hyperplane with outer normal u. Then $P = Q + R$ implies $P_u = Q_u + R_u$.

Now consider any strong chain $F_0, F_1, \dots, F_k$ of indecomposable faces of P, and let $G_0, G_1, \dots, G_k$ be the corresponding chain of faces of its summand Q. Since each $F_j = P_u$ for some u, and since F_j is indecomposable, it follows that $G_j = \lambda_j F_j + t_j$ for some $\lambda_j \geqq 0$ and $t_j \in \mathbf{E}^d$. Because $\dim(F_{j-1} \cap F_j) \geqq 1$ for each j, we then see that $\lambda_{j-1} = \lambda_j$ and $t_{j-1} = t_j$. Hence, for any strongly connected family $\mathcal{F}$ of indecomposable faces of P, there are a $\lambda \geqq 0$ and a $t \in \mathbf{E}^d$ such that, if G is the face of Q corresponding to $F \in \mathcal{F}$, then $G = \lambda F + t$.

By the hypothesis of Theorem 2, there exists such a family $\mathcal{F}$ which touches every facet of P. If $F_i = P_{u_i}$ is such a facet, then there is a vertex p of F_i in some face of $\mathcal{F}$. So, if q is the corresponding vertex of $g_i = Q_{u_i}$, we have $q = \lambda p + t$, and so the support parameters $\eta_i = \langle p, u_i \rangle$ and $\zeta_i = \langle q, u_i \rangle$ satisfy

$$\zeta_i = \langle q, u_i \rangle = \langle \lambda p + t, u_i \rangle = \lambda\eta_i + \langle t, u_i \rangle.$$

We conclude that $Q = \lambda P + t$, as we wished to show.

3. Example and remark

We have just one example here to demonstrate the greater efficacy of our new criterion. If P and Q are two polytopes in $\mathbf{E}^d$, we call the convex hull $\operatorname{conv}(P \cup Q)$ the direct *join* of P and Q if $\dim \operatorname{conv}(P \cup Q) = \dim P + \dim Q + 1$, and denote it by $P \circledV Q$. The following result is, of course, well known.

THEOREM 3. *A join of two polytopes is indecomposable.*

Every facet of the join $P \circledV Q$ contains either P or Q. So, if p, q are vertices of P, Q respectively, then the edge $E = \operatorname{conv}\{p, q\}$, which is indecomposable, meets every facet of $P \circledV Q$. By Theorem 2, with $\mathcal{F} = \{E\}$, $P \circledV Q$ is itself indecomposable.

We may observe that Theorem 2 can be generalized to a certain extent to arbitrary compact convex sets; the crucial feature is the existence of a strongly connected family of indecomposable faces which meets every one of a minimal set of support hyperplanes which determines the set. So, for example, the join of any two compact convex sets is also indecomposable.

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